Living in a contradictory world: categories vs sets

Pierre CARTIER

Abstract

In the present time, the ambition to offer global foundations for mathematics, free of ambiguities and contradictions, covering the whole spectrum of the mathematical activities, has been challenged by known flaws induced by the use and abuse of "big" categories. Unless we are ready to abandon a large part of fruitful trends in mathematical research, we have to face head on the reality (or nightmare) of contradictory mathematics.

Synopsis

In their pioneering paper on "Natural transformations", Eilenberg and Mac Lane stressed the importance of a new kind of constructions, now known as *functors*. So far the known constructions in geometry would associate two classes element by element, for instance a circle in a plane and its center. Examples coming from topology were of a different kind associating globally to a space another space (like the loop space) or algebraic invariants (like homotopy or homology groups). Also the question was raised of the naturalness of some transformations, like the identification of a finite dimensional vector space with its dual (not natural) or its bidual (natural). The insistence on transformations leads to a style of proof, which is "without points". Again, in his axiomatic description of the homology groups of a group (or Lie algebra) as given in the 1950 Cartan Seminar, Eilenberg considers a "construction" which to a group G associates the homology groups $H_i(G; Z)$ for instance. But he is not explicit about how to express such a construction in the accepted paradigm of set theory. In Cartan-Eilenberg book, there is also a description "without points" of the direct sum of two modules. So, *in the minds of the founding fathers of category theory, this theory was a kind a superstructure on the existing mathematics*, more at the level of metamathematics.

In his epoch-making Tohoku paper, Grothendieck reversed this trend. Inspired by the work of Cartan and his collaborators on sheaves and their cohomology, Grothendieck introduced head on *infinitary methods in category theory.* His purpose was to use direct limits to define the stalks of sheaves in a categorical way, since one knew already many examples of sheaves in a category, like sheaves of groups, of rings, etc Also one of the greatest discoveries of Grothendieck in this paper is the *existence of injective objects* in a reasonable category (satisfying axiom AB 5*). Going back from this abstract level, one can freely use injective sheaves, thereby greatly simplifying the general theory of sheaves.

In so doing, Grothendieck was combining two lines of thought : the rather metamathematical (hence finitary) methods of Eilenberg and Mac Lane, with the infinitary methods of Bourbaki Topology and Algebra focusing on infinite limits (direct or inverse) and universal problems. This mariage was extraordinarily fruitful for mathematics, but a price had to be paid. Categorical reasoning was "proofs without points" but the new methods required to consider the actual (not potential) totalities of all spaces, or all continuous transformations between spaces. Immediately, the old ghosts of the set-theoretic paradoxes resurfaced, like the Burali-Forti antinomy of the set of all sets, or the Richard antinomy bearing on definable objects. A natural development led to fundamental notions, like limit of a functor, representable functor and Yoneda lemma, adjoint pair of functors. But the logical disease remained, leading for instance to a questionable proof of the general existence of an adjoint functor.

If category theory can easily be formulated within a framework of first-order logic (and this led to Lawvere formulation of set theory in this spirit), and if set theory received a proper axiomatization as the Zermelo-Frenkel system, the combination of both proved explosive. Some cures were attempted, like the use of *universes* by Grothendieck and Gabriel-Demazure. But this is highly artificial, like all methods using a universal domain, and brings us to the difficult (and irrelevant) problems of large cardinals in set theory.

At the moment, the situation is not unlike the one prevailing in the 18th century in the infinitesimal calculus. Everyone knew that the existence of infinitesimal was questionable and that its use leads easily to contradictions. Today, we know about the dangerous spots, where not to swim, and try to stay away while continuing our exploration.

In my talk, I shall review a number of technical options, using type theory instead of set theory, using various sorts of axiomatic systems, or relying on topoi. At the philosophical level, the central question is the time-honored debate between *actual vs potential* infinitary entities. So it is really a question about the *ontology* of mathematics. I shall also comment on the possible use of paraconsistent logics.

Generality versus unity in geometry

Colin McLARTY

Abstract

We often hear that recent mathematics aims for maximal generality. We especially often hear it of Grothendieck's work. Deligne made it more more nuanced. He said Grothendieck worked in "the greatest natural generality," But even this formulation provoked Grothendieck to protest, pointedly, that he sought not generality, but unity. The two goals are distinct, although they have gone surprisingly far together. I will explore that distinction, and the themes of naturality, generality, and simplicity as illustrated in some particular steps in the development of Grothendieck's general idea of "space."

Euler, Maxwell, Grothendieck, and the nature of space

F. William LAWVERE

Abstract

Variable quantities such as energy and volume, metrics and affine connections, wave functions and probabilities, etc. are the ingredients of physical constitutive relations for matter and for the ether. I want to try to clarify the nature of the plenum over which these quantities vary. The default model of such a plenum has been assumed to be a topological space, but Maxwell had observed that physics needs a variety of levels of precision that can be cranked up or cranked down as required. Grothendiecks great work of 50 years ago laid the framework for a systematic use of homological algebra in complex analysis, and in so doing pointed the way not only to an explosion in the development of category theory, but also to an explicit nonlinear theory of the panorama of kinds of space implicit in mathematics. There are at least two very different classes of toposes. An oversimplification of the history of topos theory suggests that it was the extension by Grothendieck from variable truth values (open subsets) to variable discrete sets (sheaves) (permitting generalizations like individual etale schemes), which provided typical examples. However, at least as typical was his 1960 treatment of the category of all analytic spaces; the objects of any one of that ilk of topos may serve as spaces for some geometrical, physical, combinatorial, or computational purpose. These toposes have very distinctive properties that I attempt to make explicit. The method I use is based on Cantors starting point, the contrast between cohesive spaces he called Mengen and qualitatively less cohesive ones. The adjoint functors embodying such a contrast have special properties (for example a Nullstellensatz and a product-preserving left adjoint). Rational mathematical definitions of apparently philosophical terms (quality, extensive quantity, form, substance) can be given in this context and used to compare examples. A class of examples of such contrast arises by recognizing an infinitesimal space T that represents the indispensable tangent bundle functor. (This T is the concentrated definition of motion in general: a map with domain T steps into the river in one place but simultaneously not only in that place). The contrasting noncohesive spaces can then be defined as those S for which all maps from T to S are constant. Calculus and differential geometry arise from the interplay of T and the map-space construction; the real one-dimensional continuum itself arises, with its intrinsic multiplication, as a retract of the tangent bundle of T; this vindicates Eulers long-deprecated definition of reals as

ratios of infinitesimals.

From Klein to Kan: the algebra of space and the space of algebra

Jean-Pierre MARQUIS

Abstract

In their 1945 paper, Eilenberg and Mac Lane made an explicit reference to Klein's Erlanger program and sketched how they thought category theory provides a generalization of the main ideas of that program. In this talk, we will examine closely their claim about invariance of concepts. As it turns out, at the time they made this claim in 1945, Eilenberg and MacLane simply did not have the right concepts to articulate the extension properly. The required concept – that of adjoint functor– was finally introduced by Daniel Kan in 1958. We will examine how Kan came to that concept through his work on combinatorial homotopy theory and how it made possible the re- connection of category theory to Klein's program and the rigorous clarification of that connection

Grothendieck's Universe

Mihaela IFTIME

Abstract

I shall talk about Alexandre Grothendieck's life, his philosophy of mathematics, his Tohuku paper, and how his viewpoints revolutionized completely people's thoughts about many aspects of mathematics. As an example, I shall talk about the Grothendieck-Riemann-Roch theorem.

What does quantum gravity tell us about the nature of spacetime?

Louis CRANE

Abstract

We examine the classical continuum, and explore how ideas from quantum gravity connect with matematical constructions in which the point set foundation for the continuum is replaces with categorical structures.

An axiomatic approach to Einsteins vacuum field equations

Gonzalo E. REYES

Abstract

The General Theory of Relativity rests on the following fundamental assumptions:

1. Space-time is a 4-dimensional curved manifold.

2. Free falling particles describe geodesics (straightest lines) in this manifold.

3. Matter and curvature interact according to Einsteins field equations.

To obtain his Field Equations, Einstein starts from the field formulation of Newtons gravitational law, namely, the Poissons equation

$$\triangle\varphi = 4\pi K\rho$$

where φ is the gravitational potential, ρ the density of matter and K a constant, and looks for an analogue of the Laplacian operator in the relativistic context of space-time. He assumes that the analogue of the potential is the metric tensor g_{ij} and requires that the analogue of the Laplacian should be a tensor containing only first and second order derivatives, just as the ordinary Laplacian. In this respect, his approach is rather formal. An analogue of the right hand side of Poissons equation is also provided, but we will limit ourselves to the vacuum and start from Laplaces equation (previous equation with $\rho = 0$). A somewhat different approach is provided by Sachs and Wu in their book *General Relativity for Mathematicians*. They imagine that you are sitting in a free falling lift with a swarm of apples very close to you. Then Laplaces equation may be reformulated as the statement: The average of the relative acceleration of the apples with respect to you is zero. This reformulation is attractive, since analogues of notions such as velocity and acceleration should be readily found in space-time, thus providing a more physical or kinematical motivated deduction than Einsteins rather formal one.
I start from this reformulation of Sachs and Wu, but my development differs considerably from theirs. In particular, I dont assume a metric, but only a connection (or parallel transport). Furthermore I work in the context of Synthetic Differential Geometry. I formulate axioms and derive Einsteins vacuum

field equations in this context. The existence of infinitesimals in this theory, in particular, allows one to formulate mathematically the colloquial reformulation of Laplaces equation and simplify some arguments. Finally, I provide smooth topos models where the axioms hold.

Internal relativity

Olaf DREYER

Abstract

I will talk about a recent proposal of mine for a quantum theory of gravity, called internal relativity. In this approach we change the relationship between spacetime and matter. Currently we view matter as propagating on spacetime. In internal relativity, on the other hand, matter and spacetime cease to exist as distinct entities, rather, they arise simultaneously from an underlying quantum system. It is through the emergent matter degrees of freedom that geometry is inferred. We have termed our approach internal relativity to stress the importance of looking at the system from the point of view of an internal observer. We show that special relativity is then a natural consequence of this viewpoint. We also show that the presence of a massive object implies curvature. In particular we show that Newtonian gravity arises in the appropriate limit.